IMPURITY IN CONTEMPORARY MATHEMATICS

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Abstract. Purity has been recognized as an ideal of proof. In this paper, I consider whether purity continues to have value in contemporary mathematics. The topics (e.g., algebraic topology, algebraic geometry, category theory) and methods of contemporary mathematics often favor unification and generality, values that are more often associated with impurity rather than purity. I will demonstrate this by discussing several examples of methods and proofs that highlight the epistemic significance of unification and generality. First, I discuss the examples of algebraic invariants and of considering a mathematical object from several different perspectives to illustrate that the methods used in contemporary mathematics favor impurity. Then I consider an example from category theory which demonstrates how unification and generality are related to impurity and that impure solutions can be explanatory. In light of this discussion, we see that purity only has marginal value within contemporary mathematics which instead prioritizes the epistemic values associated with impurity.

1. Introduction

Throughout the history of mathematics, there have been mathematicians who have advocated for purity as an ideal of proof. That is, mathematicians have recognized the epistemic value of providing a solution to a theorem that remains within the same topic. In the 20th century mathematics changed significantly, both as a result of the introduction and development of new areas of mathematics (e.g., algebraic topology, algebraic geometry, category theory) and the rapid progression within mathematics. For this reason contemporary mathematics looks quite different from the mathematics of a century ago. Part of this mathematical progression has included an emphasis on unification and generality. Contemporary mathematics favors results that are general and able to unify different areas of mathematics. Unification and generalization go hand in hand with impure solutions, and so the importance of these epistemic

Forthcoming in *Notre Dame Journal of Formal Logic*.

ideals has often outweighed the value of purity. Consequently, purity does not play the role in contemporary mathematics that it has historically been thought to play. In other words, though purity is a valuable feature of proof, it is one that only has marginal value within contemporary mathematics, because the methods favored by contemporary mathematicians naturally give rise to impure solutions rather than pure ones.

In this paper, I will unpack the idea that contemporary mathematics tends to give rise to impure solutions and will highlight the connection between unification, generalization, and impurity. By considering several examples, we will see that the epistemic features associated with impurity are prominent within contemporary mathematics, and as a result, pure solutions are not a priority. In section 2, I will introduce the notion of topical purity, and discuss some of the different ways that it has been conceived. Then, in section 3, I will focus on the topics popularly studied in contemporary mathematics and will present some examples which raise worries about the possibility of a pure proof within these topics. In section 4, I will unpack the point that the impure methods of contemporary mathematics provide explanations by providing general and unifying solutions. After the discussions in section 3 and 4, we will have enough evidence to conclude that the importance of unification and generalization for contemporary mathematics marginalizes the value of purity.

2. Topical Purity

In the history of mathematics there have been major figures, such as Aristotle and Bolzano, who have advocated for pure methods on the basis of their epistemic value. Here I will consider three different ways of formulating *topical purity*: Aristotelian purity, Bolzanian purity, and axiomatic purity. The general idea behind topical purity, which underlies these three formulations, is that the topic of a theorem restricts the means by which you can provide a pure proof. In other words, the resources to be used in a pure proof are determined by the topic of the theorem. As put in Detlefsen & Arana (2011):

> We say that a solution ε of $\mathscr{P}$ is topically pure when it draws only on such commitments as topically determine $\mathscr{P}$ (Detlefsen & Arana (2011), pg. 13).

The most important aspect for determining purity then becomes the topical determination of a theorem. The three formulations of topical purity I discuss here differ in the ways that they account for the determination of a topic of a given theorem.

The first kind of purity I discuss is what I call *Aristotelian purity.*

> **Aristotelian purity:** A proof is pure if and only if it is contained in the same mathematical domain as the corresponding theorem.

This formulation is derived from the following assertion found in Aristotle: "[W]e cannot in demonstrating pass from one genus to another. We cannot, for instance, prove geometrical truths by arithmetic" (Aristotle (1994), 75b1-75b12). So, according to Aristotle an arithmetic proof of an geometric theorem is not a proof at all. The topic of a theorem must belong to the domain in which the theorem is stated. This separation results from Aristotle's account of the sciences. He characterized each science as a genus which consisted of properties specific to that particular science. Thus, the properties of one genus could not be applied to another genus unless it was the case that this second genus also had the property. A geometric theorem belongs to the domain of geometry, and so a pure proof must be geometric. On this view, then, the domains of mathematics are what constitute mathematical topics.

The second way of formulating purity is what I call *Bolzanian purity*, which is derived from remarks made by Bolzano.

> **Bolzanian purity:** A proof is pure if and only if it relies only on the terms used in the statement of the theorem.

This kind of purity can be interpreted in a radical way or a moderate way. The radical sense is most loyal to Bolzano, who believed that the ideal proof is purely analytic. This radical interpretation, then, requires that a pure proof only unpack the terms used in the theorem. Such a proof should not introduce terms or methods that are not analytically contained in the statement of the theorem.

Bolzano endorsed this interpretation because of his metaphysical commitments concerning justification. He believed that a proof should reveal the grounds of the theorem. These metaphysical ideas and commitments are not commonly endorsed by philosophers of mathematics today, and will not be the focus of this paper. For this reason, I will not give much attention to this radical interpretation and will instead focus on a more moderate interpretation.

The more moderate interpretation constitutes the last formulation of purity, which I will call axiomatic purity. This formulation allows for the topic to be determined by the terms of the theorem in a broader sense than analyticity.[1]

> **Axiomatic purity:** A proof is pure if and only if it relies on the theory or theories which describe the subject(s) of the theorem.

[1]It should be noted that axiomatic purity can be seen as a broadening of Bolzanian purity. Even though I will focus on the former in the following sections, the points I raise concerning axiomatic purity can also be raised concerning Bolzanian purity.

The subject of the theorem determines the theory or theories that are able to be used in a pure proof. This interpretation, or a version of it, is advocated by Mic Detlefsen and Andy Arana who describe topical determination as follows:

> [A] pure proof or solution is one which uses only such means as are in some sense *intrinsic* to (a proper understanding of) a theorem proved or problem solved (Detlefsen & Arana (2011), pg. 1).

Here the requirement that a pure proof remains "instrinsic" to its corresponding theorem does remind one of the analyticity requirement presented by Bolzano, but Detlefsen and Arana's account allows for the class of pure solutions to be broader than the class of analytic solutions. So, a pure proof is restricted to the topic that is instrinsic to the theorem.[2]

To get a better sense of this kind of purity, it is helpful to consider an example. Detlefsen & Arana (2011) presents the example of the infintitude of primes problem — the result that states that there are infinitely prime numbers. They determine the topic of this theorem as Peano arithmetic. The content of this theorem is a statement about prime numbers, and the topic of the prime numbers is Peano arithmetic. As a result of this topical determination, Detlefsen & Arana (2011) denies that Furstenberg's topological proof of this theorem is a pure proof. It is worth noting that Furstenberg's proof is purely topological in the sense that it is wholly contained in topology. So since the statement of the theorem is not a topological statement, this proof does not qualify as pure. Once the topic of a theorem is determined, the requirement of a pure proof constrains the means and methods that can be used in proving the theorem.

Now that I have introduced these formulations of purity, I will discuss the motivation for pure solutions. Purity has been thought of as an ideal (or valuable feature) of proof, but it is

[2]For more details on this account see Arana (2008), Arana (2009), Detlefsen (2015).

important to be clear about the way that purity is valuable. In short, the value of purity is seen to be epistemic, and more specifically explanatory. Detlefsen (2015) makes this point as follows:

> [A] pure proof is not, sheerly by dint of its purity, a better justification of the theorem it proves. It is, however, a more effective instrument for gaining further knowledge. We see here, then, a suggestion of the idea that purity generally increases the effectiveness of divisions of epistemic labor based on specialization. (Detlefsen (2015), pg. 191)

So the idea is that a pure proof is able to expand our knowledge beyond mere justification. One way that an pure proof can expand our knowledge is by providing an explanation in addition to a justification. This is the kind of expansion of knowledge that we will focus on here. Detlefsen (2015) also points out that Bolzano recognized this value of purity:[3]

> In truth, though, avoidance of circularity was not Bolzano's only reason for advocating purity. Indeed, it was not his principal reason. More fundamental was his acceptance of the traditional distinction between two types of reasoning in mathematics and in science generally. These were (i) confirmatory reasoning, or reasoning which convinces *that*, and (ii) reasoning which reveals the objective reasons for truth.. . . Bolzano pointed to the earliest instances of proof as his inspiration. Thales, he said, did not settle for knowledge *that* the angles at the base of an isosceles triangle are equal, though this was doubtlessly evident to him. Rather, he pressed on to understand *why*. In doing so, he was rewarded by an extension of his knowledge. Specifically, he obtained knowledge of those truths that implicitly underlay common-sense belief in the theorem. (Detlefsen (2015), pg. 185)

For Bolzano, pure proofs extend our knowledge by identifying the underlying reasons why a theorem is true, and, in this sense, they provide an explanation. This point is also made by Curtis Franks, who identifies three benefits of purity discussed by Bolzano, one of which is that pure proof reveals the grounds of the truth of the theorem (Franks (2014), pp. 7-8). This indicates that Bolzano was interested in the explanatoriness of pure proofs.

[3]Bolzano's remarks that are referred to in the following passage can be found in Bolzano (1804).

Aristotle also thought that the purpose of demonstration was explanation.[4] Purity plays an important role because an explanation is constituted by the "middle step" between the premises and the conclusion. Given Aristotle's account of the sciences, an explanation cannot come from a genus other than the one which the truth being explained belongs to. So, this middle step or explanation must be pure in the sense that it belongs to the same domain (or genus) as the premises and conclusion. Again, we see the close connection between purity and explanation.

The key to this line of thinking is that purity is taken to be epistemically significant in the sense that it extends knowledge by giving us not only knowledge that but also knowledge why. Knowledge that is usually associated with justification and knowledge why with explanation. Then pure methods are epistemically valuable in that they provide explanation in addition to justification. In short, pure proofs have a tendency to be explanatory proofs.

Given this correlation between purity and explanation, it seems clear that purity should be taken to be an ideal of proof. But this correlation or ideal is only useful when purity arises somewhat naturally or makes sense in the given situation. I will argue that purity does not arise naturally or make sense in large parts of contemporary mathematics. As a result, purity does not seem to be as epistemically valuable in the contemporary setting as it has been historically. Whatever the value of purity may be, that value appears to be at odds with the ambitions that drive a considerable portion of current mathematical research. Given the changes in subject matter (or topic) in contemporary mathematics, purity is no longer a priority. Further, the impure methods found in contemporary mathematics produce generality and unification and, in this sense, are epistemically valuable and produce new mathematical insights. In the next

[4]For a detailed discussion of Aristotle's account of demonstration see Mendell (1998).

two sections, I will fill out this argument. First, in section 3, I will argue that the topics of contemporary mathematical research embrace the impure rather than the pure. Second, in section 4, I will argue that the impurities that center contemporary mathematics have provided their own ways of extending knowledge and providing explanation.

3. Topics in Contemporary Mathematics

Transfers from one domain of mathematics to another pervade contemporary mathematics. In fact, many areas of contemporary research are centered on such transfers — e.g., algebraic topology, algebraic geometry, and category theory. When introducing the notion of homology, a focus of algebraic topology, Eilenberg and Steenrod describe the topic as follows:

> [A] homology theory is an *algebraic image* of topology. The *domain* of a homology theory is the topologist's field of study. Its *range* is is the field of study of the algebraist. Topological problems are converted into algebraic problems (Eilenberg & Steenrod (1952), pg. vii, their emphasis).

Mac Lane expresses a similar sentiment:

> Homology provides an algebraic "picture" of topological spaces, assigning to each space X a family of abelian groups $H_0(X), \ldots, H_n(X), \ldots$, to each continuous map $f : X \to Y$ a family of group homomorphisms $f_n : H_n(X) \to H_n(Y)$. Properties of the space or the map can often be effectively found from properties of the groups H_n or the homomorphisms f_n (Mac Lane (1963), pg. 1).

Both of these passages suggest that the transfer from topology to algebra provides a way of highlighting topological properties. It is important to realize that both of these descriptions recognize that topology and algebra are distinct domains of mathematics. But it is precisely the impure jump from topology to algebra that allows us to progress in algebraic topology. That is, we need to make this jump from topology to algebra in order to study such notions as homology and homotopy.

In addition to algebraic topology, algebraic geometry relies on the transfer between algebraic properties and geometric properties. Category theory allows for the transfer from a given topic of mathematics to almost any other. That is, a category itself is a way of formally capturing a domain or topic of mathematics, and the existence of functors between categories provides a formal way of transferring information from one category to another — i.e., from one topic to another. In short, there are several areas of research in contemporary mathematics that specifically aim at encoding some mathematical information in terms of other mathematics. The pervasiveness of such topical transfers gives rise to the question of whether pure methods are possible in contemporary mathematics. In this section, I will discuss some cases where results in contemporary mathematics are resistant to purity.

To clarify the idea of topical transfer in contemporary mathematics, I will discuss the example of algebraic invariants in algebraic topology. These invariants encode topological properties within algebraic objects (most commonly, groups) and also preserve these properties under certain transformations. For instance, any two topological spaces, whose corresponding algebraic objects disagree — i.e., are not isomorphic — do not themselves agree — i.e., are not homeomorphic.[5] So, inequivalence at the algebraic level gives rise to inequivalence at the topological level. Homology, mentioned above, and homotopy are both examples of algebraic invariants. So, two spaces whose homology groups are not isomorphic will not themselves be homeomorphic.

[5]It is important to note that though invariants tell us when two spaces are not equivalent, they do not tell us when two spaces are equivalent. That is, two spaces whose corresponding algebraic objects are isomorphic are not necessarily homeomorphic.

These algebraic invariants are the basis of algebraic topology. The goal of algebraic topology is to study the topological properties of a space using algebraic tools. This transfer from geometry or topology to algebra makes the subject matter — i.e., the mathematics — simpler and more clear.[6] The fact that the goal of algebraic topology relies on this combination of two distinct topics in mathematics seems to preclude pure methods. Strictly speaking, the study of algebraic topology is a study of topology. A result in algebraic topology is about some topological property and advances our knowledge of topology. But the means by which we advance our topological knowledge is algebraic. We rely on algebraic constructions and facts to draw conclusions about topological properties. The topic of algebraic topology, therefore, relies on the combination of two distinct topics — i.e., topologyand abstract algebra — and in this way it seems to be impure itself.

To see this impurity more clearly let's consider an example.

Theorem 3.1. *The fundamental group of* $\mathbb{R}$ *is trivial — i.e.,* $\pi_1(\mathbb{R}, 0) \cong 0$.[7]

Before discussing the proof of this result, notice that on the face of it this is a theorem about two particular groups. Namely, it states that these two groups, $\pi_1(\mathbb{R}, 0)$ and 0, are isomorphic. This suggests that the advocate of axiomatic purity would identify group theory as the topic of this result. But really, it is a result about the topological properties of $\mathbb{R}$ — specifically, the contractibility of $\mathbb{R}$. Perhaps, upon further consideration, the axiomatic purist would recognize that the statement of this theorem is not purely group theoretic. In particular, it could be said

[6]For more details on this point see McLarty (2006) which discusses the specific cases of homology groups and their introduction to mathematics.

[7]Here 0 denotes the trivial group — i.e., the group under addition consisting of the additive identity element alone.

that the use of π_1 in the statement of the theorem allows for the use of homotopy in a pure solution. To better consider this example, let's consider the proof. It is fairly simple and goes as follows:[8]

Proof. First note that we can define the following homotopy: $k : \mathbb{R} \times I \to \mathbb{R}$ such that $k(s,t) = (1-t)s$. This is a homotopy from the identity to the constant loop at the base point, c_0. Let $f : I \to \mathbb{R}$ be a loop with basepoint at 0. Then we can define $h(s,t) = k(f(s),t)$. Then h is also a homotopy from f to c_0 and shows that f is homotopy equivalent to c_0. So any based loop at 0 in $\mathbb{R}$ is homotopy equivalent to c_0 — i.e., can be deformed to c_0. Hence, $\pi_1(\mathbb{R}, 0) \cong 0$. □

So, this is a straightforward proof in the sense that it simply shows that any loop in $\mathbb{R}$ is homotopic to the constant loop via a linear homotopy. The proof solely relies on talk of homotopy, a topological concept. It proves a topological result — namely that $\mathbb{R}$ is contractible — but one that is stated in algebraic terms. To determine whether this is a topically pure proof we must determine which topic it belongs to: topology or algebra. But it does not seem to wholly belong to one or the other. Rather, it belongs to the combination of the two. Hence, it is a result of algebraic topology. Is this proof purely algebraically topological? Maybe. But, if so, why is it not the case that any algebraic or topological result can be proved with appeal to the other and still count as pure — i.e., purely algebraically topological. Historically, pure methods excluded those that combined algebraic and geometric reasoning, and I imagine that the same would have to hold for combinations of topological and algebraic reasoning.

[8]This version of the proof is paraphrased from May (1999), pg. 8.

Moreover, the whole point of this sort of mathematics is, not just to make topological discoveries, but to try to uncover hidden relationships between these topics. To even appreciate the spirit of the result and what mathematicians have discovered, you have to recognize a sense in which algebra and topology are distinct topics, and that this result uncovers a connection between them. That connection is part of the mathematical discovery, and one that is overlooked if, in the allegiance to an ideal of purity, one tries to ignore the historical topical boundaries. In this way, it seems that the goals and focus of algebraic topology undermine the ideas behind purity.

Despite it's impurity, this proof is epistemically valuable — more specifically, it is explanatory in the sense that it provides a satisfying answer to the question of why the theorem is true. This proof explains why it is that $\mathbb{R}$ is contractible.[9] Namely, because any loop can be deformed into the constant loop via a linear homotopy, $\mathbb{R}$ can be shrunken to a point. In a bit more detail the argument goes as follows: The proof given above shows us that any two loops with common base points are homotopic — i.e., they can be deformed into one another. In particular, if we take one of these loops to be the constant loop at the base point, then we see that any loop (with this same base point) can be deformed to this constant loop.[10] So again, the fact that $\mathbb{R}$ is contractible is explained by the fact that there exists a linear homotopy

[9] More accurately, this proof explains *in part* why $\mathbb{R}$ is contractible. That is, not all topological spaces with trivial fundamental group are contractible. For instance, the 2-sphere, S^2, has trivial fundamental group but cannot be contracted to a point. The 2-sphere fails to contract to a point essentially because it is "hollow" — when we situate S^2 in R^3, it encloses points in R^3 that are not in S^2. So, $\mathbb{R}$ is contractible not only because it has trivial fundamental group, but also because it is not "hollow" in the way that the 2-sphere is. Thank you to an anonymous referee for pointing out the need to be more careful when discussing the relation between having trivial fundamental group and being contractible.

[10] Again, this will only result in contractibility if the space is one that fails to be "hollow".

between any loop and the constant loop at its basepoint. This shows us that the impurities in contemporary mathematics do not diminish the possibility of explanatory proofs.[11]

Now that we've worked out the details of this example, we can consider how it is viewed by the different formulations of purity. The above discussion illustrates that Aristotelian purity is not viable in contemporary mathematics. Recall that Aristotelian purity considers a pure proof one that remains within the mathematical domain of the theorem. The example of algebraic invariants raises a problem for this notion of purity, because it requires the combination of two distinct domains. The result proved above states that two particular groups are isomorphic, but is proven purely topologically. This occurs because the statement, though algebraic, encodes information about the topological properties of $\mathbb{R}$ — it encodes the fact that $\mathbb{R}$ is a contractible space. This translation between topological and algebraic terms and methods constitutes a transfer between different domains of mathematics and, in this way, violates Aristotelian purity. Since it is central to topics such as algebraic topology, algebraic geometry, and category theory, to use such domain transfers, Aristotelian purity is not a viable possibility in much of contemporary mathematics.

The discussion so far does not completely respond to axiomatic purity, though it does raise some doubts. Again, the example discussed above gives a theorem stated in terms of groups — $\pi_1(\mathbb{R}) \cong 0$. So, it appears that the content of this theorem consists of groups and that a pure proof in the axiomatic sense would rely only on group theory. As noted earlier, it can be argued that there is more to the content of the theorem than merely groups. The subject of the theorem is a specific kind of group — the fundamental group — and this kind of group has topological content. Thus, there is reason to think that a proof that relies on homotopy,

[11]This point will be discussed in more detail in the next section.

like the one above, could be considered pure in the axiomatic sense. The fact the the subject of the theorem is the fundamental group makes both homotopy and group theory available as resources for a pure proof.

To further consider how axiomatic purity fits into contemporary mathematics, let's look at another example: Brouwer's fixed point theorem.

Theorem 3.2. *Any continuous function $f : D^n \to D^n$ for $n \in \mathbb{N}$ has a fixed point.*[12]

Proof. To show that this holds, it suffices to show that there is no retraction from the disk to the boundary of the disk.[13] For the sake of contradiction, suppose that there is a retraction $r : D^n \to \partial D^n$. Then the composition with the inclusion map $i : \partial D^n \to D^n$ is the identity on ∂D^n. That is,

$$\partial D^n \xrightarrow{i} D^n \xrightarrow{r} \partial D^n = id_{\partial D^n}.$$

This composition induces the following composition in homotopy:

$$\pi_1(\partial D^n) \xrightarrow{i_*} \pi_1(D^n) \xrightarrow{r_*} \pi_1(\partial D^n) = id_{\pi_1(\partial D^n)}.$$

[12]This proof can be found in both May (1999) and Riehl (2016).

[13]A retraction is a map such that when composed with the inclusion map, the composition is the identity map. That is, a retraction $r : D^n \to \partial D^n$ would be such that

$$\partial D^n \xrightarrow{i} D^n \xrightarrow{r} \partial D^n = id_{\partial D^n}$$

where $i : \partial D^n \to D^n$ is the standard inclusion map. If there were no fixed point for the function $f : D^n \to D^n$, then we would be able to define a retraction $r : D^n \to \partial D^n$ that sends $x \in D^n$ to the intersection of ∂D^n and the ray starting at $f(x)$ and passing through x. Note that this retraction requires that there is no fixed point, since if there were a fixed point, we would not be able to draw the ray necessary for defining the retraction. So, showing that there exists no retraction implies that there is a fixed point for f.

But since the D^n has trivial fundamental group and $\partial D^n = S^{n-1}$ has cyclic fundamental group, we get the following contradiction:

$$\mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z} = id_{\mathbb{Z}}.$$

In other words, the induced maps in homotopy imply that the identity map on $\mathbb{Z}$ is trivial. So, we see that there cannot be a retraction from D^n to ∂D^n, and, consequently, that there is a fixed point for f. □

In terms of the purity of this proof, it seems mostly pure until we consider the induced maps between homotopy groups. That is, it seems reasonable to identify the content of the theorem as topological, in the sense that we are able to use the theory given by the axioms defining a topological space. To use a phrase of Detlefsen & Arana (2011)'s, a "proper understanding" of the theorem identifies the topic as being topology. There is nothing in the statement of the theorem that suggests the need to consider anything outside of point-set topology. After all, the result is about continuous maps between topological spaces. The problem then seems to be that the contradiction needed for the proof relies on the transfer to algebra. The incompatibility of the homotopic properties of the disk and the homotopic properties of the sphere do not become apparent until we transfer to algebraic terms. It is the algebraic fact that a nontrivial homomorphism cannot factor through the trivial group by means of a surjection followed by an injection. This move may be considered to belong to algebraic topology rather than algebra more broadly, but it certainly extends past the axiomatic theory of topological spaces. As a result, this move is a breach of purity — it requires methods outside of the topic of the theorem.

Now, if we take the content of the theorem to be topology in a sense broader than point-set topology then one might argue that this proof is actually pure. That is, if we take the content of the theorem to be topology, and we take this to mean that any topological theory is available to us, including algebraic topology, then we would be able to use homotopy while staying within the bounds of purity. This, however, seems to broaden the bounds of purity too far. Recall that this kind of purity calls for a closeness between the statement of the theorem and its solution. If we allow for everything that falls under the umbrella of topology to count as close to this theorem, then closeness loses much of its significance. The desire for closeness is at least in part the result of a desire to avoid distractions. If we are proving a theorem about continuous functions between n-disks, then the need to appeal to homotopic properties seems to be a distraction in the sense that our understanding of these continuous functions does not appeal to or require an understanding of homotopic properties. If we allow for all of algebraic topology to be considered close to all of point-set topology, then we are allowing for such distractions.

The purpose of this discussion has not been to conclusively conclude that algebraic topology is wholly impure, but rather to raise some points that indicate that the methods of algebraic topology are largely resistant to purity. There are at least some clarifications that need to be made by advocates for axiomatic purity in order to make sense of how we should determine the content of a theorem in cases such as these. Moreover, these examples highlight a method of reasoning crucial for contemporary mathematics, one that seems to stand in tension with purity. Further, this use of algebraic invariants represents a more general phenomena. Mathematics *often* yields many perspectives on the same thing, and the ability to draw connections and relations between these perspectives is of great interest and significance. For instance, it is a significant discovery that we are able to think about topological properties in algebraic terms.

Another example of this phenomena is the Gauss-Bonnet theorem which relates geometric and topological properties.[14] More specifically, it concludes that a geometric property (Gaussian curvature) can be understood in terms of a topological property (Euler characteristic). In short, this result states that the total curvature of a space, M, is given by $2\pi\chi(M)$ where $\chi(M)$ is the Euler characteristic of the space. More formally, the theorem is stated as follows:

Theorem 3.3 (Gauss-Bonnet)**.** *Let M be a compact, 2-dimensional Riemannian manifold. Then*

$$\int_M K + \int_{\partial M} k_s ds = 2\pi\chi(X).$$

I will not discuss the proof of this result, but even from the statement of the result we see that it relates different mathematical perspectives. In contemporary mathematics, this is thought to be a significant achievement. In fact it is clear that the whole interest in this result arises from the surprising fact that these geometric and topological properties, which had separate and intricate histories, turn out to be so systematically related. When you understand this result as having a single topic, rather than as explicitly relating two distinct topics (so that one of its proofs could be understood as pure), it is hard to understand also what its mathematical significance is — something plain to any mathematician or student of mathematics.

This way of relating different mathematical perspectives differs slightly from the algebraic invariants case discussed above. In the above discussion, we saw that there was a transfer from

[14]It would be incorrect to say that Gauss and Bonnet are contemporary mathematicians, but regardless this result is important for the contemporary study of differential geometry.

one area of mathematics to another for the purpose of introducing clarity and simplicity to the study of the first area. In the case of the Gauss-Bonnet theorem we are not transferring from one topic to another in the same way. Rather, we are recognizing that we have the ability to study compact, 2-dimensional Riemannian manifolds from many different perspectives — two of which are geometry and topology — but that, despite this ability, we are still studying the same thing. So, the geometric properties of these manifolds are not wholly distinct from the topological properties. This is a significant discovery precisely because of the unification it introduces.

A call for purity counters this interest in unification. Purity limits the resources and perspectives available for a solution and, as a result, it will limit the ability to draw connections and highlight relations between different mathematical perspectives. The Gauss-Bonnet theorem demonstrates the importance of realizing these connections and relations, but it does not highlight the limitations of purity. To demonstrate the limitations of purity, we need to consider a result in which the relation of different mathematical perspectives appears within the proof rather than the statement of the theorem. Such an example will be presented in the next section. The goal of the next section will be to expand on the points made here. First, to give an example where mathematical perspectives are related within a proof, and second to show that this method of reasoning — one that emphasizes unification — is epistemically valuable and has, in fact, become a priority of contemporary mathematics.

4. Unification and Impurity

In this section, I want to discuss the way that impure methods have proved to be epistemically valuable and want to highlight that these methods have been more useful in contemporary mathematics than pure methods. First, recall that one reason cited for privileging pure methods is that a proof using pure methods is able to extend our knowledge past mere justification. One way of interpreting this is that proofs using pure methods provide explanations of why their theorem is true. The methods that are preferred in contemporary mathematics also provide explanations, but in different ways. Pure solutions provide explanations by zeroing in on the subject of the theorem, but contemporary mathematics privileges explanations that clarify a bigger picture. To makes sense of this, it is best to consider an example.

The example I will focus on is an instance of a specific kind of example that has become popular in contemporary mathematics. This kind of example consists of a category theoretic result of which there are corollaries in other areas of mathematics. Category theory is a powerful area of mathematics which enables us to capture and describe different mathematical domains and theories, while also providing a way of relating these domains and theories. In fact, category theory has interestingly explained some connections and parallels between results that are otherwise difficult to account for. In some cases, category theory is able to relate results that seem, on the face of it, to have nothing in common. The significance of these kinds of results is their applicability across different areas of mathematics which results in explanatory unification. Consider the following three results:

Theorem 4.1. *For any function $f : A \to B$, the direct image,*

$$f_* : \mathscr{P}(A) \to \mathscr{P}(B); X \mapsto f(X)$$

preserves unions, and the inverse image,

$$f^{-1} : \mathscr{P}(B) \to \mathscr{P}(A); X \mapsto f^{-1}(X) = \{\, a \ \mid\ f(a) \in B \,\}$$

preserves both unions and intersections.

Theorem 4.2. *For any vector spaces, U, V, W,*

$$U \otimes (V \oplus W) \cong (U \otimes V) \oplus (U \otimes W).$$

Theorem 4.3. *The free group on the set $X \sqcup Y$, i.e., the disjoint union of sets X and Y, is the free product of the free group on X and the free group on Y. That is,*

$$F(X \sqcup Y) \cong F(X) * F(Y).$$

The first of these results (theorem 4.1) belongs to basic set theory, the second (theorem 4.2) belongs to linear algebra, and the third (theorem 4.3) belongs to abstract algebra. So, a pure proof of any of these results would have to rely on methods within these topics or related theories. But, there is a proof, for each of these results, relying on category theory that is both impure and explanatory. In fact, it is the impurity and the resulting unification that makes these proofs explanatory.

These proofs rely on the following category theoretic theorem:[15]

Theorem 4.4. *Left adjoints preserve colimits. And dually, right adjoints preserve limits.*

Each of these results can be shown simply by verifying that the objects used in the proofs are adjoint functors and (co)limits. For instance, here is the proof of theorem 4.1 when treated as a corollary of this category theoretic result:

[15]It is worth noting that this is not an exhaustive list of the corollaries of this category theoretic results. There are at least two more corollaries, one in set theory and one in homological algebra. I will not present them here, but I recommend that an interested reader refer to chapter 4, section 5 in Riehl (2016) for more details on these additional corollaries as well as the ones I've mentioned in the body of the paper.

Proof. First, it is important to realize that $\mathscr{P}(A)$ and $\mathscr{P}(B)$ are categories since they are partially ordered under the subset relation.[16] Then, notice that unions are colimits and intersections are limits in the category $\mathscr{P}(A)$. Further, f^{-1} and f_* can be understood as functors between the powerset categories. In particular, f_* is a left adjoint functor, and f^{-1} is both a left adjoint functor and a right adjoint functor. Then by applying theorem 4.4, we see that f_* preserves unions and f^{-1} preserves both unions and intersections. □

So, essentially, this result follows immediately from the fact that the direct image, inverse image, intersections and unions have this category theoretic structure. The other results — i.e., theorem 4.2 and theorem 4.3 — also follow similarly from this category theoretic result.

The proofs of these results that treat them as corollaries of the category theoretic theorem are impure. That is, the statement of the results do not make any mention of category theory, and so, the reliance on category theoretic structure to prove the corollary violates axiomatic purity. Instead of zeroing in on the properties specific to the subject of the theorem, we broaden our perspective to the category theoretic level. We move to a more general result in order to see why and how it is that the specific result holds. This move to the general is what results in the impurity of the solution.

The impurity introduced by treating these results as corollaries provides us with epistemically significant solutions. In particular, the resulting solutions are explanatory — in the sense that they provide an answer to a why-question. If I asked why is it the case that the direct

[16]Note that for any set A, $\mathscr{P}(A)$ is a category since it is a partially ordered set under the subset ordering. More specifically, the objects of the category are subsets of A and the arrows are given such that there exists an arrow $X \to Y$ if and only $X \subseteq Y$.

image preserves unions or why the tensor product distributes over the direct sum, an acceptable answer would be to say that these results hold because of the category theoretic structure underlying each of these constructions. That is, these results hold *simply because* they have this category theoretic structure. It is because the direct image is a left adjoint functor that it preserves unions, which are colimits in the powerset category. And the tensor product distributes over direct sum because $U \otimes -$ is a left adjoint functor and $\oplus$ is the coproduct. So, the category theoretic proofs provide explanations of these more specific results despite the breach in purity.

Moreover, these solutions are explanatorily valuable because of their ability to unify. The fact that these three, seemingly unrelated, results can all be proved in essentially the same way is a significant discovery. There is epistemic gain when we realize that these distinct corollaries can all be traced back to the same theorem. We see that there are similarities between union, direct sum, and disjoint union that otherwise evade us. In their respective contexts, these operations are all playing the same role — namely the role of colimit in their given category. This similarity unifies the different topics in such a way that seems to put purity at a disadvantage. Remaining pure and within the scope of a single topic would prevent such similarities from surfacing. But these similarities help to better our understanding by highlighting comparisons between different mathematical topics. So, if I am considering the direct sum of vector spaces, I can use the fact that it is a structural analog to the union of sets in order to better make sense of it. Comparison across mathematical topics can provide useful insight. In this way, purity seems to limit our mathematical knowledge. As a result, impure proofs have something to offer that pure proofs cannot — namely, unification across mathematics.

This example is not unique; such examples are a strength of category theory.[17] That is, category theory has the ability to unify different areas of mathematics and, in doing so, it highlights surprising mathematical relations and similarities of mathematical constructions. The impurity of category theory gives us a global perspective that we are unable to achieve from the localized, pure point of view. The values of unification and generality are not specific to category theory, but instead central to contemporary mathematics as a whole. Results that relate different areas of mathematics or provide a general perspective are of great interest to working mathematicians.

In introducing his approach to algebraic topology, Peter May describes a general approach to constructing homology theories as both "conceptual and illuminating" (May (1999), pg. 1). It is the generality that May commends. The generality gives a far-reaching perspective that still enables us to improve our conceptual understanding. Dennis Sullivan attributes the interest in providing a classification of manifolds to the "intertwining" of their geometric and algebraic properties (Sullivan (1974), pg. 1). He is interested in establishing connections between and thereby unifying the geometric and algebraic properties of manifolds. Emily Riehl identifies an advantage of category theory as the fact that it "provides a cross-disciplinary language for mathematics designed to delineate general phenomena, which enables the transfer of ideas from one area of study to another" (Riehl (2016), pg. ix).

[17]For more examples of this kind see Riehl (2016).

These expressions from prominent mathematicians highlight the interest in general and unifying approaches for contemporary mathematics. May's book presents many ways of approaching standard questions in algebraic topology with general approaches which provide more insight into the given topic and related topics. Sullivan's main interest (in the paper mentioned above) lies in the entanglement of geometry and algebra that arises when we classify manifolds. The main interest of the paper is impurity, since to approach these classification results with an emphasis on purity would be to miss out on the insight into the relationships between algebra and geometry. Riehl recognizes that the value of category theory is its ability to characterize mathematics across disciples, again emphasizing the ability to unify.

These comments are not meant to serve as an argument for the idea that we should prefer impurity to purity, but instead are meant to highlight the fact that in contemporary mathematics interest lies in those methods and solutions that provide generality and unification. Thus, I am not interested in arguing against the value of purity here. Instead, my aim is to point out the fact that contemporary mathematics has been more focused on characteristics related to impure methods, such as generality and unification, rather than pure methods.

To make this point more clear, let's consider more closely the epistemic benefits of pure versus impure methods. The epistemic significance of purity results from the fact that it allows you to focus on the content of a theorem when providing its solution. This focus allows us to become more familiar with the properties of the object and, as a result, we get a better grasp of the object. So, pure proof allows us to become familiar with the specific details of the subject of the theorem. That is, we are able to zero-in on the subject without distraction.

Impure proofs, on the other hand, rely on "distraction", but in such a way that turns out to be epistemically valuable. To clarify this, let's return to the example above. Suppose that I am interested in showing that the direct image preserves unions. Then the turn to a category theoretic theorem appears to be a distraction, but in fact it provides an explanation of the more specific result. The general perspective introduced by category theory introduces new tools to our epistemic tool-box. Whereas both purity and impurity have their epistemic advantages, contemporary mathematics favors the impure.

5. Conclusion

In this paper, I have argued that impure methods are prevalent in contemporary mathematical research and that these methods provide their own epistemic advantages. I have used the topic of pure methods to motivate my discussion, but I have not concluded that pure methods fail to be epistemically advantageous. Rather, the topics in contemporary mathematics have made impure methods more natural ways of providing solutions and have become the standard. Moreover, the generality and unification that accompany these impure methods often produce surprising and significant results. It is an advantage when a mathematical result is able to highlight the connections that hold between different areas of mathematics. As mathematics continues to expand, the ability to formally relate different corners of mathematics will continue to be of value, and the concern for such relations will continue to emphasize an interest in generality and unification. As we have seen, the epistemic advantages of impure methods differ from those of pure methods, and though both have their advantages, within the setting of contemporary mathematics the impure is usually more valuable.

Acknowledgments

I would like to thank Curtis Franks, Tim Bays, Colin McLarty, and an anonymous referee for their comments on various drafts of this paper. I would also like to thank the audiences at the Notre Dame Graduate student colloquium and at the *Mathematics in Philosophy: Purity and Idealization* conference, and Paddy Blanchette and Anand Pillay for organizing this special issue. Lastly, I would like to thank Mic Detlefsen for introducing me to the topic of purity in mathematics and to many other topics within the philosophy of mathematics.

References

Arana, A., 2008, "Logical and semantic purity". *Protosociology*, 25, 36–48.

Arana, A., 2009, "On Formally Measuring and Eliminating Extraneous Notions in Proofs". *Philosophia Mathematica*, 17, 189-207.

Aristotle. (1994) *Posterior Analytics*, (trans.) J. Barnes, Oxford University Press.

Bolzano, B. (1804). "Preface to Considerations on Some Objects of Elementary Geometry" in *From Kant to Hilbert vol. 1*, 1996, (ed.) W.B. Ewald. Oxford University Press, 172-174.

Detlefsen, M. & Arana, A., 2011, "Purity of Methods", *Philosopher's Imprint*, 11(2), 1-20.

Detlefsen, M. (2015) "Purity as an Ideal of Proof" in *Philosophy of Mathematical Practice*, (ed.) P. Mancosu, New York: Oxford University Press, 179-197.

Eilenberg, S. & Steenrod, N., 1952,*Foundations of Algebraic Topology*, Princeton, New Jersey: Princeton University Press.

Franks, C., 2014,"Logical Completeness, Form and Content: an archaeology", in J. Kennedy (ed.) *Interpreting Gödel: Critical Essays*. Cambridge: Cambridge University Press, 78-106.

Mac Lane, S., 1963, *Homology*, Berlin: Springer.

May, P., 1999, *A Concise Course in Algebraic Topology*, Chicago: University of Chicago Press.

McLarty, C., 2006, "Emmy Noether's 'Set Theoretic' Topology: From Dedekind to the Rise of Functors" in *The Architecture of Modern Mathematics*, (eds.) J. Ferreirós & J. Gray, Oxford University Press, 187-208.

Mendell, H., 1998, "Making Sense of Aristotelian Demonstration", in C.C.W. Taylor (ed.) *Oxford Studies in Ancient Philosophy*, vol 16., Oxford: Clarendon Press, 161-225.

Riehl, E., 2016, *Category Theory in Context*, Dover Publications Inc.

Sullivan, D., 1974, "Genetics of Homotopy Theory and the Adams Conjecture", *Annals of Mathematics*, 100(1), 1-79.