MATHEMATICAL EXPLANATION IN PRACTICE

Abstract. The connection between understanding and explanation has recently been of interest to philosophers. Inglis & Mejía-Ramos (2019) propose that within mathematics, we should accept a functional account of explanation that characterization explanations as those things that produce understanding. In this paper, I start with the assumption that this view of mathematical explanation is correct and consider what we can consequently learn about mathematical explanation. I argue that this view of explanation suggests that we should shift the question of explanation away from why-questions and towards a "what's going on here" question. Additionally, I argue that when we recognize the connection between understanding and explanation then naturally see how more than just proofs can be explanatory. I expand this point by detailing how definitions and diagrams can be explanatory. In all, we see that when we take seriously the connection between understanding and explanation, we get a better sense of how explanation arises within mathematics.

Keywords: Philosophy of Mathematics, Mathematical Practice, Explanation, Understanding, Definitions, Diagrams, Contemporary Mathematics

1. Introduction

The relationship between understanding and explanation has been of recent interest to epistemologists and philosophers of science alike. Recently, Inglis & Mejía-Ramos (2019) has proposed an account that highlights the significance of such a connection within the realm of mathematics. In particular, Inglis & Mejía-Ramos (2019) apply Wilkenfeld's account of functional explanation to the field of mathematics.[1] One advantage of this view is that it makes clear why we are and should be interested in explanation—namely, because it leads us to understanding. Many mathematicians have expressed the significance of understanding within the practice of mathematics and for the successful progress of mathematics. Perhaps most well-known by philosophers, William Thurston identifies understanding to be what we aim to achieve when we do mathematics (Thurston (1995)). Saunders Mac Lane also identifies the significance of understanding for mathematics when he says:

> [T]he progress of Mathematics depends on a counterpoint between solving conundrums and searching for the concepts which provide better understanding (Mac Lane (1986), p. 38)

This recognition of the role of understanding makes the idea of functional explanation in mathematics particularly compelling.

In this paper, I will start from the assumption that mathematical explanations produce understanding and will consider what we can consequently learn about explanation. In section 2, I will argue that this perspective on explanation suggests that within the realm of mathematics we should shift focus away from why-questions, around which explanation is usually thought to revolve, and instead consider the "what's going on here" question. To make this case I will consider examples from basic trigonometry, the arithmetization of analysis, and virtual knot theory. Then, in section 3, I will consider how the characterization of explanation in terms of understanding broadens the class of what can count as an explanation. In particular, I will make the case that both definitions and diagrams can be explanatory and will again use examples to unpack these points. From the discussions had in sections 2 and 3 we will see that when we recognize the connection between explanation and understanding we are able to better account for how explanation arises within mathematical practice—in particular, we will see that mathematical explanation is more than just answers to why-questions and explanatory proofs.

2. The Question of Mathematical Explanation

In this section, I propose that we consider the question "what's going on here" rather than why-questions when we seek mathematical explanations. One common aspect of traditional accounts of explanation is that they approach the topic of explanation in terms of why-questions. In the context of mathematics, this amounts to taking explanatory proofs to be ones that answer the question "why does this result hold?" Some prominent accounts of mathematical explanation that consider explanatory proofs from this point of view include those of Mark Steiner, Philip Kitcher, and Marc Lange.[2] Each of these accounts has a different way of determining when an answer to a why-question gives a satisfying explanation with Steiner emphasizing characterizing properties, Kitcher emphasizing unification, and Lange emphasizing the exploitation of a salient feature of the result, but each aligns with the framework that explanatory proofs are those that show us *why* a theorem is true, rather than simply *that* it is true. It is this framework that it is so ubiquitous throughout the discussion of mathematical explanation and that requires that explanatory proofs be understood as answers to the question: "why is this result true?" The specific accounts I have mentioned have received a lot of philosophical attention and each have received some criticism, but in this paper, I would like to focus on a criticism of the general framework of mathematical explanation as answers to why-questions that underlies these accounts.[3]

The crux of my criticism of this why-question framework is that it does not capture all that there is to mathematical explanation.[4] In particular, these why-questions do not always capture the understanding component of explanation, at least not in the case of mathematics.[5] For instance, consider the following question: *why does the polynomial* x^2+2 *have no real roots?* Now a reasonable and perfectly satisfying answer to this question is that the discriminant of this polynomial is negative. This is a perfectly good answer, but this answer does not provide us with or lead us to an understanding of why the polynomial has no real roots. In particular, the answer indicates that there is an understanding that there is some connection between the discriminant and the roots of the polynomial, but it does not suggest an understanding of the connection itself. In short, the answer to the why-question, given above, is not a satisfying answer to the "what's going on here" question because it still leaves us wondering what the connection is between the discriminant and the roots of the polynomial. An answer to the "what's going on here" question would lead us to an understanding of why the polynomial $x^2 + 2$ has no real roots.

The main idea behind this call to change the question of explanation in mathematics is that the question of explanation should be one whose answers naturally produce understanding. As we see from the example of the discriminant, answers to why-questions do not always produce understanding. In order to illustrate that answers to the "what's going on here" question do produce understanding, it is important to first clarify what I have in mind by understanding. The following passage from David Hilbert nicely gets at what I have in mind.

> [T]he tendency toward *intuitive understanding* fosters a more immediate grasp of the objects one studies, a live *rapport* with them, so to speak, which stresses the concrete meaning of their relations.
>
> As to geometry, in particular, the abstract tendency has here led to the magnificent systematic theories of Algebraic Geometry, of Riemannian Geometry, and of Topology; these theories make extensive use abstract reasoning and symbolic calculation in the sense of algebra. Notwithstanding this, it is still as true today as it ever was that *intuitive* understanding plays a major role in geometry. And such concrete intuition is

> of great value not only for the research worker, but also for anyone who wishes to study and appreciate the results of research in geometry (Hilbert & Cohn-Vossen (1952), iii).

It is through explanations that we are able to achieve the "intuitive understanding" and "live rapport" that Hilbert mentions here. When doing mathematics, you are not merely partaking in formal reasoning but you are trying to get a sense of the objects that you study. This amounts to developing a familiarity with the mathematics and it is this familiarity that constitutes understanding.

To get a clearer picture of what familiarity is, it is helpful to discuss why familiarity plays an important role in mathematics. In short, the answer is that the abstract subject matter of mathematics calls for a way of making mathematical objects accessible. Our understanding of mathematics becomes deeper the more and more familiar we become with mathematics and mathematical objects.

For instance, consider the abstractness of the complex numbers and complex analysis. The introduction of imaginary numbers was for the purpose of solving cubic polynomials, and people were initially skeptical of these numbers and their mathematical status. Imaginary numbers continued to be used because of their practical usefulness—they allowed us to solve problems (particularly, cubics).[6] This skepticism really corresponds to a lack of familiarity, precisely because imaginary numbers are quite abstract. In order for a field to develop around imaginary numbers, there needed to be a way of accessing these abstract numbers and this was eventually accomplished. As Kitcher (1984) describes:

> An important episode in the acceptance of complex numbers was the development, by Wessel, Argand, and Gauss, of a geometrical model of the numbers.... Proponents of complex numbers had ultimately to argue that the new operations shared with the original paradigms a susceptibility to construal in physical terms. The geometrical models of complex numbers answered to this need, construing complex addition in

> terms of the operation of vector displacement and compelx multiplication in terms of the operation of rotation. (Kitcher (1984), 176).

The skepticism surrounding the complex numbers when they were first introduced indicates that the mathematical community lacked an answer to the "what's going on here" question. But, the work of Wessel, Argand, and Gauss helped to provide an answer to this question by making the complex numbers more familiar to us. This was done primarily by putting the complex numbers in familiar, geometric terms. Once mathematicians were able to reason about the complex numbers in these familiar terms, they had the understanding necessary to develop complex analysis. Thus, it was only when the "what's going on here" question was answered that mathematicians had an understanding of the complex numbers. (This example will be discussed in more detail in Section 3).

Here I have used the notion of familiarity to describe the kind of understanding that I have in mind, but it is worth acknowledging that the concept of understanding is philosophically rich and that there are many different kinds of understanding.[7] Given the aim of this paper, I intend my characterization of understanding in terms of familiarity to be a form of explanatory understanding, though in some examples it may also seem that objectual understanding is produced. This gives rise to the question of how, on my account, the two notions (i.e., explanatory understanding and objectual understanding) are related in the context of mathematics. This is an interesting question and one worth pursuing, though it is outside of the scope of the current paper.[8] For the remainder of this paper I will continue to use my characterization of understanding in terms of familiarity without explicitly commenting on the relationship between explanatory and objectual understanding.

Now, to further illustrate the significance of this "what's going on here" question, I will unpack some examples. When working in mathematics, either in a research or a classroom setting, this question naturally arises. For instance, suppose you are presented with the fact that $sin(60°) = \frac{\sqrt{3}}{2}$. Most students are aware of this fact because they have memorized the unit circle and the corresponding trigonometric values at $30°, 45°$, and $60°$. Of course, this method of memorization is not very illuminating. A student who is more interested in understanding the trigonometric functions will likely wonder "what's going on here?" And there is a satisfying answer to this question. Trigonometric functions fundamentally represent the relations of sides in right triangles. So, $sin(60°)$ is making reference to the relationship between the side opposite the $60°$ angle and the hypotenuse in a 30-60-90 triangle. And this fact, together with some facts about 30-60-90 triangles, can be used to derive the fact that $sin(60°) = \frac{\sqrt{3}}{2}$.

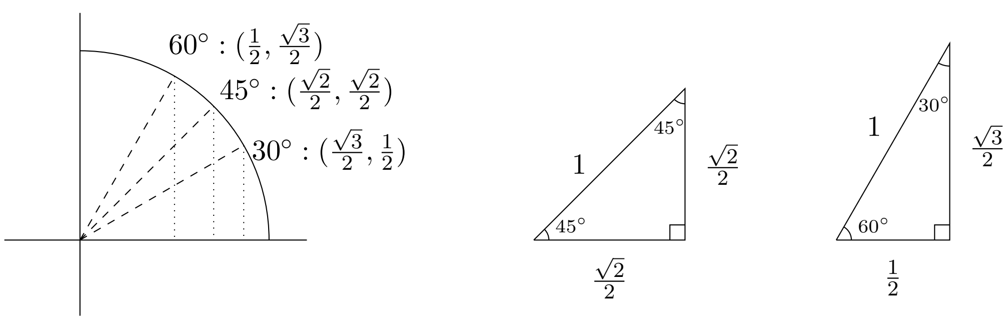

FIGURE 1. Here we see the first quadrant of the unit circle and the triangles corresponding to $45°$ and $60°$.

As in figure 1, we can draw a 30-60-90 triangle contained in the unit circle whose hypotenuse forms a $60°$ angle with the x-axis. Then we can use the fact that the sides of the of a 30-60-90 triangle stand in certain relations to one another (given in figure 1) to conclude that the side opposite the $60°$ angle has length $\frac{\sqrt{3}}{2}$. This use of triangles explains what is going on behind the statement that $sin(60°) = \frac{\sqrt{3}}{2}$ by highlighting the fact that the sine function represents the relation of the opposite side and the hypotenuse in a right triangle.

In some instances, the "what's going on here?" question will align with the why-questions that are often associated with explanation. For instance, in the trigonometry example, we might have asked "why is $sin(60°) = \frac{\sqrt{3}}{2}$?" and ended up with the same answer. But the "what's going on here?" question applies to a wider range of cases. In particular, in research settings the "what's going on here" question proves to be crucial for successfully progressing within mathematics. To see this, let's now consider an example of how this question arises in a research setting. I will discuss an example that is familiar to the philosophical literature: the discovery of nowhere differentiable continuous functions. This example is often discussed by philosophers of mathematics in connection to the issue of standards of rigor. But this example also interestingly illustrates the importance of the "what's going on here?" question.

In the 19th century, as mathematicians were making a lot of progress in analysis, a need for clearer notions of continuity and differentiability arose. In the beginning of the 19th century, mathematicians conflated the notions of continuity and differentiability. It was believed that any continuous function was also differentiable, at least at all but finitely many points.[9] For instance, the absolute value function is continuous and is differentiable at all points except $x = 0$. So it was known that the two notions come apart, but it was not realized exactly how far apart these notions come. But as the 19th century progressed the distinction between these notions came to light.

This development was driven by the discovery of examples of continuous functions that failed to be differentiable at infinitely many points.[10] These examples further separated the notions of continuity and differentiability. Bolzano first discovered a function that was continuous but that was not differentiable on countably many points. Then, Weierstrass discovered an example of a continuous function that was nowhere differentiable. These discoveries shocked the mathematical community and gave rise to the question "what's going on here?" Continuity and differentiability were thought to go hand in hand, but with Weierstrass's example, we see that a function can be continuous and wholly fail to be differentiable—the two notions completely come apart.

In light of this example, it was recognized by the mathematical community that there was a need to get clearer about what was going on in cases of continuous nowhere differentiable functions. As Manheim (1964) puts it, this example "precipated a crisis which caused mathematicians to re-examine the construction of the real number system" (Manheim (1964), 76). The philosophical significance of this re-examination is usually described as a call for a higher standard of rigor and less reliance on intuition. And the arithmetization project did certainly introduce rigor, but in doing so, it also made us more familiar with the real numbers and functions defined on them. To see this more clearly, let's consider in more detail what is learned from these examples of continuous functions that are nowhere differentiable.

Intuitively, a function is continuous when you are able to draw it without picking up your pencil. And a function is differentiable when it is smooth, when it lacks sharp corners or cusps.[11] If we again consider the absolute value function, then we intuitively see that it is continuous because we can draw it without picking up our pencil, and we can see that it fails to be differentiable at $x = 0$, because the point at $x = 0$ forms a corner since it is the point where the function sharply changes from decreasing to increasing. The example Weierstrass discovered is a continuous function that is constantly turning in the way that the absolute value function does at $x = 0$. This example tells us that it is possible to have a function that can be drawn without interruption (i.e., without picking up ones pencil) but that will fail to be smooth. It is worth mentioning, that in practice it is impossible to draw Weierstrass' function, which relies on the use of limits.[12] His example describes how to systematically add corners to the function and then uses limits to continue this process to result in the desired nowhere differentiable function. Thus, in theory, drawing this function would never require one to pick up there pencil but it also would not be smooth at any point.

The examples discovered in the 19th century highlighted this distinction between theory and practice, which in turn highlighted the fact that intuition was insufficient for getting a complete understanding of what is going on with the real numbers. To get this complete understanding, it is necessary to consider examples that can only be realized in theory and to rely on limits to bring these examples to light. This feature of analysis was only realized once examples that countered our intuitions were discovered and the project of arithmetizing analysis was pursued.

In this example of continuous nowhere differentiable functions, there does not seem to be a corresponding why-question. Perhaps, we could ask "why are there examples of continuous nowhere differentiable functions?", but this doesn't seem to be what we are really interested in. We are interested in getting a clearer picture of the real numbers, one that highlights which properties give rise to such examples. The "what's going on here?" question allows us to develop an understanding of the real numbers and functions defined on them. In hindsight, it was clear that such an understanding was lacking from intuition alone making it necessary to further explore what is going on.

To further clarify the role of the "what's going on here" question, it is useful to consider one more example. This example is taken from more recent mathematical developments within knot theory. Mathematically, a knot is a closed curve in $\mathbb{R}^3$ homeomorphic to the circle, S^1. These curves contain a finite number of crossings, places where two strands meet and one strand goes over the other. An important component of knot theory is finding ways to represent knots so that their properties can be studied. One way of representing knots is through knot diagrams, which results from projecting the knot onto a plane leaving a break in the curve to represent the under-strand of a crossing. Another way is by means of Gauss codes, which encode information about the crossing of the knot. Figure 2 gives an example of a knot, in its knot diagram form.

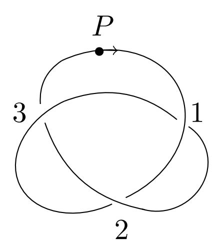

FIGURE 2. Here the crossings are labeled 1, 2, and 3, the starting point used to determine the Gauss code is labeled P, and the orientation of the knot is indicated by the arrow.

The Gauss code corresponding to this knot diagram is given by $O1{+}U2{+}O3{+}U1{+}O2{+}U3{+}$. To obtain this code, we start at P and use the orientation given by the arrow to traverse the knot, recording crossing information as we come to each crossing. Doing so, the first crossing we come to is 1 and we pass through it via the over-strand. Thus, the first two characters of the Gauss code are "$O1$". The third character in the code is "+" because crossing 1 has positive orientation, determined using the right hand rule. This process is continued until we arrive back at the point P at which point our Gauss code will include a triple for the over- and under-strands of each crossing.

This shows how you can obtain a Gauss code from a given knot diagram, and you can also recreate a knot diagram when given the Gauss code. But interestingly, not every string that follows the structure of a Gauss code corresponds to a knot. So, if you randomly generated a string of triples of the right form (e.g., "O1+") you would not be guaranteed a corresponding knot diagram. So, the question of what is going on when a string of triples does not correspond to a knot diagram naturally arises.

To become more familiar with this phenomenon, let us now consider an example. Consider the following Gauss code: $O1+O2+U1+U2+$. Note that this string of characters does have the structure of a Gauss code as described above. But when we try to draw the corresponding knot diagram we run into a problem. Figure 3 shows the step by step process of attempting to draw the corresponding knot diagram.

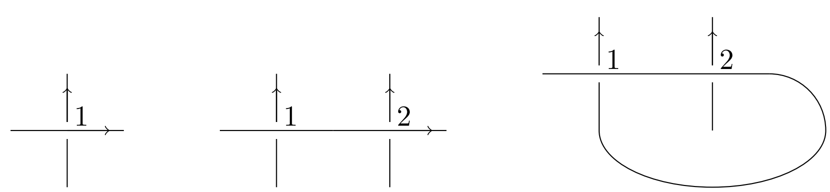

FIGURE 3. Here we show step by step how the knot diagram would be drawn from the given Gauss code. The arrows indicate the orientation of the knot and insure that each crossing has the correct sign.

We are able to draw the first three triples of the Gauss code, i.e., $O1+O2+U1+$, but after this we encounter a problem. The next step should connect the loose end from the under-strand

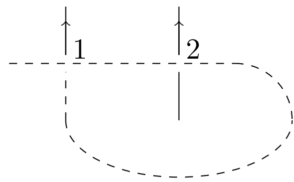

FIGURE 4. Here the dashed strand illustrates the obstacle in continuing on to the under-strand of 2 crossing. In order to reach this under-strand, we must cross through this dashed strand.

of the 1 crossing to the under-strand of the 2 crossing, but we cannot access this under-strand because it is enclosed by another strand. That is, we have to cross through the dashed strand in figure 4 in order to connect to the under-strand of the second crossing while still preserving the correct orientation. But, the Gauss code does not include a third crossing so we cannot include one in the knot diagram. At this point we have an answer to question, *why are do some Gauss codes not correspond to knot diagrams?* Some Gauss codes present crossing information in an order that cannot be realized without additional crossings. But it still is not entirely clear what is going on here. That is, there is nothing about the mathematical definition of knots or the properties of crossings that suggest that they must conform to a certain order. There is nothing in particular about the string "$O1+O2+U1+U2+$" that suggests that it cannot be realized so there is more to be considered about this situation.

Knot diagrams can be thought of as projections of the 3-dimensional knot onto a plane—similar to casting the shadow of the knot onto a wall. But we can also think about embedding the knot into surfaces other than the plane. For instance, we can embed a knot into the torus. When we do this, we get some clarity about what is going on in the situation above. Figure 5 shows a way that we can embed the trefoil knot (which is shown in Figure 2) into the torus.[13] Notice that when we embed the knot in this way there are two crossing of the usual over/under form, but that one of the crossings of the trefoil knot takes a different form. In this third "crossing", one strand goes through the top half of the torus and the other strand goes through the bottom half (the dotted strand depicts the strand that goes through the bottom half of the torus). If this knot were projected onto the plane, then these strands would form a crossing of the usual over/under form, but since the torus is a thickened surface, a crossing can appear in this new form.

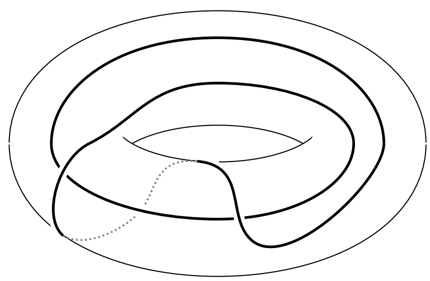

FIGURE 5. Here is a depiction of the trefoil embedded in the torus in a way that corresponds to the Gauss code $O1+O2+U1+U2+$.

This new form of crossing, first introduced in Kauffman (1999), is called a virtual crossing and is depicted in knot diagrams by a circled crossing. So we can represent the knot corresponding to the Gauss code $O1+O2+U1+U2+$ with the virtual knot diagram show in Figure 6.[14] With this introduction of virtual crossings, we can now say that every Gauss code corresponds to a virtual knot diagram—a knot diagram that may (or may not) include one or more virtual crossings.

It is with this introduction of virtual crossings that we get a satisfying answer to the "what's going on here" question. We see that there is more mathematical structure captured by the Gauss codes that is simply not captured by the classical knot diagrams (i.e., knot diagrams that do not include virtual crossings). The introduction of virtual knot theory not only allows

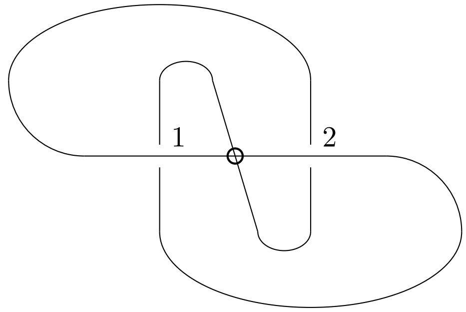

FIGURE 6. This is the virtual knot diagram corresponding to the Gauss code $O1+O2+U1+U2+$ where the circled crossing indicates a virtual crossing.

us to establish a one-to-one correspondence between Gauss codes and virtual knot diagrams, but it also provides new ways of studying both classical knots and virtual knots. For instance, virtual knots give rise to invariants that allow us to compare two knots to determine when they differ.[15] So, the consideration of virtual crossings gives us a more complete understanding of knots and their representations by knot diagrams and Gauss codes. As mentioned above, the why-question about the relationship between Gauss codes and classical knot diagrams can be superficially answered in a satisfying way but it does not give a complete understanding. That is, we have a satisfying answer to why some Gauss codes do not correspond to classical knot diagrams when we see that some Gauss codes correspond to a knot diagram that requires more crossings than are identified by the Gauss code. But this answer does not clarify what is going on and does not promote an understanding of what happens when a Gauss code does not give rise to a classical knot diagram.

These examples describe situations in which it is natural to wonder what is going on to produce observed phenomena. But more than only showing how this question arises, these examples also illustrate how the question is answered. When we use special right triangles to demonstrate what is going on with the sine function and its evaluation, we become more familiar with what the trigonometric functions represent. With this realization we are able to easily derive the evaluations of the trigonometric functions without relying on memorization. In the example of the arithmetization of analysis, spurred by the discovery of continuous nowhere differentiable functions, we become more familiar with the real numbers and the properties of functions defined on the real numbers. In particular, we become more familiar with what it means for a real-valued function to be continuous and differentiable. And by introducing virtual crossings into knot theory we develop a better understanding of how knots are represented and how their representation presents information about their crossings.

We ask the question “what’s going on here?” when we realize that we lack familiarity with the relevant mathematics. And when we are able to answer this question and gain the desired familiarity, we have an explanation. More specifically, what we are really interested in when we seek a mathematical explanation is to develop understanding by means of developing familiarity with the relevant mathematics. We are hoping to get a better sense of the mathematical entities or structures we study. We don’t simply want to know why the result holds, but instead we want to develop a picture of the relevant mathematics that makes the result hold. When we seek an explanation of the fact that every group is isomorphic to a subgroup of a symmetric group (i.e., Cayley’s theorem), we want more than just knowledge why this result holds. We want to develop an understanding of the nature of groups that makes it the case that every group is related to the symmetric groups in this way. The “what’s going on here?” question encourages us to develop this kind of understanding in a way that “why-questions” do not.

Moreover, the “what’s going on here?” question arises in environments other than just the theorem-proof environment. This allows us to widen the scope of mathematical explanation.

We are now able to consider examples of mathematical explanation that come in forms other than proof. These other forms of explanation are the focus of the next section.

3. More Than Proof

Once we recognize that mathematical explanation is not focused around why-questions, it becomes clear that explanations can take forms other than proof.[16] Here I will discuss two forms of explanation distinct from proof: definition and diagram. Though my discussion is focused on these two forms, I do not mean to suggest that these are the only non-proof forms of explanation. There are certainly others. But the discussion of definitions and diagrams is a good place to start.

First, I will consider how definitions can be explanatory.[17] In mathematics, it is common to consider whether a given definition is the *right* one. This is, at least in part, due to the fact that there are often multiple definitions of the same concept. For instance, Thurston (1995) identifies several different ways that we are able to think about the derivative. When there are so many ways of thinking about a mathematical concept, a question naturally arises: "what is the right definition of the concept?"

In Tappenden (2008), Jamie Tappenden points out that mathematicians often ask questions about which definitions are "right" or "correct" and that these questions influence the direction of mathematical research. The "right" definition introduces clarity to the concept being defined so that we are able to gain a clear idea about it. More than just introducing clarity, new definitions introduce new perspectives that can reveal mathematical truths that were not revealed by previous definitions. In these ways, definitions play an important role in influencing the direction of mathematical progress. Moreover, since definitions are our first introduction to mathematical notions, the "right" definition will be one that does a lot to make us familiar with the mathematical object being defined.

Tappenden illustrates the significance of definitions and the phenomenon of the "right" definition with the example of prime numbers. He points out that when we are first introduced to prime numbers, the given definition is that a natural number, $n > 1$, is prime if its only factors are 1 and n. But when we move on to group theory or algebraic number theory we are introduced to a different definition of prime number. In particular, a number, $a \neq 1$, is prime if whenever a divides the product bc, a divides b or a divides c.[18] In the context of the natural numbers these two definitions are equivalent, but when we consider the notion of prime number in contexts other than $\mathbb{N}$, the two definitions do not coincide. A number may be prime in the first sense but fail to be prime in the second sense. Tappenden (2008) presents the example of 2 which is prime in $\mathbb{N}$ but fails to be prime in $\mathbb{Z}[\sqrt{5}i]$ (see pp. 267-268).

Given these two definitions, we can consider which definition is the "right" one. Tappenden answers this question by identifying the second definition as the "proper one". He says:

> The reason for counting the second definition as the proper one is straightforward: The most significant facts about prime numbers turn out to depend on [the second definition]. ...The familiar school definition only captures an accidental property; the essential property is: $a|bc \rightarrow (a|b \text{ or } a|c)$ (Tappenden (2008), 268).

The second definition should be privileged because it is better suited to capture important phenomena within algebraic number theory. For instance, the applicability of the second definition extends past the natural numbers to include other structures (such as $\mathbb{Z}[\sqrt{5}i]$), which the first definition fails to do. This example and Tappenden's general point about there being "right" or "proper" definitions suggests that definitions play an important role in developing an understanding of mathematics. Some definitions are better than others, and a good definition can result in "a significant advance in knowledge" (Tappenden (2008), 269).

This discussion of the existence of multiple definitions of the same concept and the privileging (at least in certain contexts) of one definition over others nicely prepares us for the discussion of explanatory definitions. The "right" definition is accurately characterized as such often because it is explanatory.

A definition will be explanatory when it helps to answer the "what's going on here?" question. That is, when it helps us develop our familiarity with and understanding of the relevant mathematics. To illustrate how a definition can be explanatory, I will now consider the example of the circle, which can be defined in several different ways. I will consider three ways of defining circle and discuss how each definition can teach us something about the explanatory potential of definitions. First consider two ways of defining circle.

Definition 1. The set of all points in $\mathbb{R}^2$ of the same distance from a given point, i.e., the center.

Definition 2. The set of all points in $\mathbb{R}^2$, (x, y), satisfying the equation $x^2+y^2 = r^2$, for some non-negative real number r.[19]

The first of these definitions characterizes circles in the context of the real plane equipped with the usual metric, whereas the characterization provided by the second definition is in the context of algebraic geometry. That is, the second definition identifies the algebraic constraints on circles, which allows us to compare circles with other geometric objects that are also subject to algebraic constraints. This algebraic characterization is not something captured by the first definition. It is true that these definitions are logically equivalent—each implies the other—but as I will discuss in more detail later in this section, the privileging of one definition over another depends on the context of the definitions.[20] Thus, the logical equivalence of the definitions does not mean that the two definitions are equally valuable in all contexts.

Given these two definitions we can ask the question of whether either is explanatory. That is, does either one increase our familiarity with the concept of circle? The immediate answer seems to be that the first definition does. This definition formalizes our intuitions about circles. Circles are a bit different from other mathematical entities because they are something that we come across fairly often in the natural world.[21] In this way, we have encountered circles fairly often even before any mathematical endeavors. But having observed things in the natural world that are circle-shaped is not sufficient for developing a mathematical familiarity with the concept of circle. That being said, the first definition given here does increase our familiarity. It highlights the mathematical property that needs to be satisfied in order for something to be a circle—namely, that all points of the circle must be equidistant from the center. This is a property that we may be able to establish based on our observations of circles, but the definition mathematically formalizes our observations. The mathematical sophistication introduced by this formalization gives us the ability to compare circles to other mathematical concepts and entities, and in this way increases our familiarity with circles.

It is important to note that my account of explanatory definition does not require that an explanatory definition be unique. So we also need to consider whether the second definition is explanatory. This definition is a bit more abstract in that it less obviously characterizes a circle. Again, the first definition is not only written in terms of the familiar notion of distance but it is also easily accessible independent of a person's mathematical inclinations. The second definition, however, involves a little more mathematical savvy.

To determine whether this second definition is explanatory, we need to consider the context of algebraic geometry. One goal of algebraic geometry is to characterize geometric objects in algebraic terms, by their algebraic constraints. In this setting, a case may be made that this definition is explanatory. It makes circles, as objects of algebraic geometry, more familiar to us. It does this, not only by giving us an algebraic characterization of circles, but also by providing us with a characterization that allows us to compare circles with other geometric objects by algebraic means.

For instance, this algebraic characterization of the circle allows us to easily compare circles and ellipses. The algebraic characterization for ellipses is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a, b \in \mathbb{R}_{>0}$ correspond to the major and minor axes of the ellipse.[22] An ellipse is a circle when the major and minor axes are equivalent. This is easily seen from these algebraic characterizations. If $a = b$, then $a^2 = b^2$ and multiplying the ellipse equation by a^2 will yield the following equation: $x^2 + y^2 = a^2$. This is the equation for a circle with radius of length a. This comparison is something that is immediately obvious only from the second definition. This example shows how easily we can compare circles and ellipses with the definition from algebraic geometry. We can similarly compare circles to other geometric constructions by use of their algebraic characterizations. This gives the second definition preference in the context of algebraic geometry.

This example highlights two important points about explanatory definitions—that the context of the definition matters, and that explanatory definitions come in degrees. The second point is closely related to the first. Note that mathematical skill and proficiency is a matter of degree in the sense that the longer or more intensely you study mathematics, the more skillful you become. Not only do you become more skilled, but you also become introduced to and proficient in more areas of mathematics. As a result, a long-time professor will be more likely to recognize a definition as explanatory than would a student just beginning their studies in mathematics. The fact that a beginning student is unable to recognize the explanatoriness of the second definition does not at all detract from its explanatoriness.

The fact that recognition of the second definition's explanatoriness requires more mathematical skill points to the fact that this definition makes the circle familiar in a more sophisticated way. The circle becomes more familiar via the second definition because this definition highlights the algebraic properties of the circle. It reveals to us more sophisticated mathematical properties than the properties highlighted by the first definition, and consequently it makes us more familiar with circles. So, the second definition is more explanatory than the first.

Along with this picture of degrees of explanation, it is important to notice that there is an underlying progression. The first definition plays an explanatory role in a person's mathematical development before the second definition can. Although the second definition is more explanatory, the explanatoriness of the first definition still plays a crucial role in a person's mathematical development. Moreover, the fact that explanations come in degrees allows for progress and development in mathematics. There are likely undiscovered ways of conceiving of the circle and these ways may provide new explanations, ones that familiarize us with the circle in new ways. It is the discovery of such new conceptions and perspectives that shape the development of mathematics.

To account for this dependence on mathematical skill and proficiency, I will say that some explanatory definitions are more *accessible* than others. So, the first definition is more accessible than the second despite the fact that the latter is more explanatory than the former. It will often be the case that a more explanatory definition is less accessible. This is because there is a correlation between the explanatoriness of a definition and the mathematical skill required to appreciate the explanatoriness. Definitions that highlight the mathematically sophisticated properties of an entity or structure require more mathematical skill to appreciate their explanatoriness.

To further push this picture of the degrees of explanation and accessibility, we can consider a third way of defining circle:

> **Definition 3.** A circle is a topological space homeomorphic to the quotient space given by the unit interval with the standard topology under the equivalence relation that identifies 0 and 1 and identifies every other point with itself.

This definition certainly introduces more mathematical sophistication than the other two. Indeed, to even parse this definition a fairly developed mathematical background is required. But more importantly, this definition further illustrates the importance of context when considering whether a definition is explanatory.

For this definition to be explanatory, it must highlight the features and relations of circles in the context of topology. This definition informs us how to construct a circle. Specifically, it says that a circle is the result of taking the unit interval and connecting the two endpoints. This is analogous to taking an untangled piece of string and bringing the two ends together to form a closed loop. Doing this will rarely result in a perfect circle, but each resulting shape is topologically equivalent to the circle. So, this definition makes familiar to us the topological relations a circle stands in. It highlights which topological spaces are equivalent to the circle. This feature makes the definition explanatory.

This topological definition introduces a mathematical perspective on circles that is importantly different from the other two definitions. The extension of this topological definition differs from the extensions of the first two definitions, which is a result of the fact that topological equivalence is determined by homeomorphism. When a homeomorphism exists between two topological spaces, the two spaces are topologically equivalent. As a consequence, a square and a circle are topologically equivalent. A square, however, does not satisfy the two definitions given above.

So, this definition provides a new way of conceiving of the circle. Through a topological lens, the structure of the circle differs from the geometric structure. Topology is sometimes described as a generalization of geometry and this more general perspective is what accounts for the broader extension. But the intuitive notion that the topological definition aims to capture is the same as in the other two cases. It is the context that differs and results in a difference in the extension.

As a result of the change in context, this definition loses the advantages of the second definition given above, the one in the context of algebraic geometry. In particular, just as a square is topologically equivalent to the circle, an ellipse will also be. So we lose the ability to nicely contrast the circle and the ellipse. But there are advantages to the topological definition. In particular, it makes clear the topological properties of the circle and its relation to other topological spaces. As a result, this third definition is still more explanatory than the definition presented in terms of distance from the center. This not only further demonstrates the existence of degrees of explanation, but also clarifies that the degrees of explanation do not take the form of a linear order. Rather, the degrees of explanation have the structure of a tree, where two definitions can be more explanatory than a third in such a way that the explanatoriness of the two definitions is not comparable.

This discussion of different ways of defining circles illustrates the way that explanatory definitions manifest themselves. It demonstrates that a mathematical concept can have multiple explanatory definitions of varying degrees and shows how definitions play an important role in making mathematics more familiar to us. Given this example, however, it may be difficult to imagine a situation when we are introduced to a definition that is not explanatory. For this reason, I now want to discuss an example of a non-explanatory definition.

Recall that the definition of a normal subgroup is given as follows:

> **Definition 4.** A subgroup H of G is called a *normal subgroup* if for every $g \in G$ and $h \in H$, $ghg^{-1} \in H$.

This definition is clear, concise and easy to parse but it does not actually give us much information about normal subgroups or their value. In fact, the definition seems a bit arbitrary and it is not at all clear why a subgroup closed under conjugation in this way would be of interest to us. It does not answer the "what's going on here" question.

Normal subgroups are of interest to us because they allow us to construct quotient groups. Recall that a quotient group, denoted G/H, requires that H be a normal subgroup of G and consists of the left cosets (i.e., $\{ gH \mid g \in G \}$). The construction of quotient groups is significant throughout algebra and it is only with normal subgroups that we can construct groups in this way. Additionally, normal subgroups are interesting because of there correspondence to the kernels of homomorphisms. Every kernel is a normal subgroup (of the domain of the homomorphism) and every normal subgroup is the kernel of some homomorphism. This fact largely contributes to the significance of normal subgroups. But the definition of normal subgroups does not in any way suggest the significance of normal subgroups for constructing quotient groups or for studying homomorphisms by means of their kernels and as a result it does nothing to reveal what is going on with normal subgroups. It's not just that there are facts about normal subgroups that further reveal their significance, as it is often the case that theorems and propositions will further develop the significance of a mathematical concept. But in the case of normal subgroups, the definition does not in any way suggest that normal subgroups are an interesting or significant notion despite the fact that they play an important role throughout group theory.

In the above discussion, we have seen how definitions are able to increase our familiarity with mathematics. Definitions often shape our perspective on a given area of mathematics and as a result they are influential in shaping mathematics and mathematical understanding. Given the significant role that definitions play in mathematics, an account of mathematical explanation should recognize that definitions can be explanatory and do contribute to our understanding.

Another form that mathematical explanation can take is that of diagrams. The role of diagrams in mathematics has received some philosophical attention, but the discussion is usually centered around the question of whether diagrams can provide justification.[23] The focus on justification can overlook other epistemic values of diagrams, such as explanatoriness. We often introduce diagrams in mathematics because doing so increases our familiarity.

One example of an explanatory diagram is the one that illustrates the stereographic projection, shown in Figure 7. The stereographic projection gives an isomorphism between $\mathbb{C} \cup \{\infty\}$

and the sphere, S^2. This is a useful way of thinking about the complex plane, particularly when we are interested in complex functions. Conway (1978) motivates the introduction of the stereographic projection as follows:

> Often in complex analysis we will be concerned with functions that become infinite as the variable approaches a given point. To discuss this situation we introduce the *extended plane* which is $\mathbb{C} \cup \{\infty\} \equiv \mathbb{C}_\infty$. We also wish to introduce a distance function on $\mathbb{C}_\infty$ in order to discuss continuity properties of functions assuming the values of infinity. To accomplish this and to give a concrete picture of $\mathbb{C}_\infty$ we represent $\mathbb{C}_\infty$ as the unit sphere in $\mathbb{R}^3$ (Conway (1978), 9).

So $\mathbb{C}_\infty$ has importance for the study of complex functions because it provides us with a way of understanding how complex functions can take on values of infinity and provides a way of defining distance. But it is the stereographic projection that provides a "concrete picture" of what $\mathbb{C}_\infty$ is. This is emphasized when we consider what is going on in $\mathbb{C}_\infty$. That is, $\mathbb{C}_\infty$ takes the complex plane together with a point *at* infinity. But the complex plane is usually understood as $\mathbb{R}^2$, with real and imaginary axes in place of the x and y axes. This space extends towards infinity in infinitely many directions, so where exactly is this point *at* infinity?

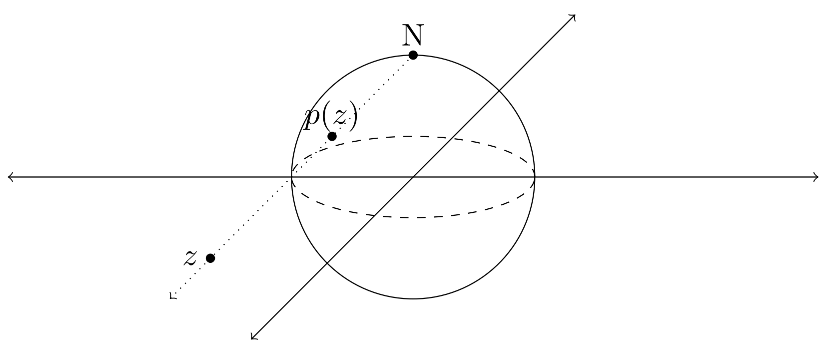

FIGURE 7. Here the sphere is given a north pole, the point at $(0,0,1)$ in $\mathbb{R}^3$. The stereographic projection maps a complex number z to the point on the sphere that is collinear with z and the north pole, N. More specifically, $z = (a,b) \mapsto \left(\frac{2a}{1+a^2+b^2}, \frac{2b}{1+a^2+b^2}, \frac{-1+a^2+b^2}{1+a^2+b^2}\right)$ and inversely, $(x,y,z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$.

It is difficult to imagine how this works when we're thinking of the complex plane in the usual way. But the stereographic projection provides us with a new way of thinking about the complex plane, specifically in terms of the 2-sphere. And this new way of thinking makes clear where the point at infinity is. It is located at the north pole of the sphere—what I have labeled N in Figure 7. It is through the stereographic projection that we become familiar with $\mathbb{C}_\infty$. Moreover, we become familiar with $\mathbb{C}_\infty$ in a way that allows us to make sense of functions that take on infinity as a value.

This diagram does not justify that the isomorphism given by the stereographic projection exists. Instead, the diagram familiarizes us with how the stereographic projection works and how we can think of the complex plane in terms of the sphere. With this diagram we are able to visualize how the complex plane can be "folded-up" to form the 2-sphere. When we do this, we also see more clearly the role that infinity plays in complex analysis. Instead of the complex plane extending towards infinity in infinitely many directions, we see all of the complex plane stretching towards one point representing infinity.

A diagrammatic representation is particularly useful in cases like the stereographic projection. The mapping that gives us the stereographic projection is difficult to wrap one's head around. It is given as follows:

$$(a, b) \mapsto \left(\frac{2a}{1 + a^2 + b^2}, \frac{2b}{1 + a^2 + b^2}, \frac{-1 + a^2 + b^2}{1 + a^2 + b^2} \right) \text{ and inversely, } (x, y, z) \mapsto \left(\frac{x}{1 - z}, \frac{y}{1 - z} \right).$$

These are not particularly nice formulas and from them alone it is difficult to visualize what the stereographic projection is doing. But the corresponding diagram amends this by giving a clear visual.

Another example of an explanatory diagram is one that Steiner appeals to in his discussion of mathematical explanation, though he does not identify the diagram itself as being explanatory. Steiner (1978) presents an explanatory proof of the result that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ that makes use of the following diagram.[24] This diagram puts the algebraic result in geometric terms and so provides a way of visualizing what is going on in the sum. The diagram, see figure 8, itself is not constitutive of the proof, but instead provides an aid that helps the reader become familiar with what is going on in the sum. This diagram illustrates that the sum, $\sum_{k=1}^{n} k$

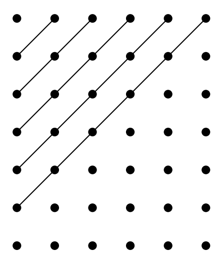

FIGURE 8. This diagram shows how two isosceles triangles whose legs are of length n can be put together to form a rectangle of dimensions $n$by $n + 1$.

corresponds to a right isosceles triangle, whose legs have length n. Once we recognize that this sum corresponds to an isosceles triangle we can think of its value in terms of the area of a rectangle. In particular, we can fit two of these isosceles triangles together to form a rectangle with dimensions $n \times (n + 1)$. Thus we can calculate the area of the triangle as $\frac{n(n+1)}{2}$. This diagram nicely complements a proof of the fact that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ because it provides a visual representation of the sum, which helps us become familiar with the sum.

Most people's first encounter with this equation is in a high school level second course in algebra, or perhaps a first course in calculus. The equation is introduced for the purposes of manipulating and evaluating series. This first introduction will rarely be accompanied by a proof or an explanation of why this equation holds, and the equation itself seems arbitrary and non-obvious at first glance. That is, the equation itself does not do much to familiarize us with the sum. There is a need for further discussion in order to really understand what is going on behind this sum. But the diagram presented here provides a way of guiding this discussion and gaining familiarity with the sum.

These two examples show that diagrams can be of value for mathematical practice, particularly when it comes to developing familiarity and seeking explanation. But we can recognize this even without the consideration of specific examples. Diagrams are often crucial components of lecture. They help to unpack what is often left out of textbooks. Moreover there is great appreciation of illuminating diagrams. Michael Spivak's *A Comprehensive Introduction to Differential Geometry* is recognized as an excellent reference for differential geometry in part because he includes many detailed (and sometimes hand-drawn) diagrams that help us to become familiar with the objects of differential geometry (Spivak (1970)). This is invaluable for a particularly abstract subject like differential geometry.

Hilbert & Cohn-Vossen (1952) is another text that is largely focused on providing diagrams that help to give the reader familiarity with mathematics.[25] In the preface to this work Hilbert identifies the value of diagrams as follows:

> In this book, it is our purpose to give a presentation of geometry, as it stands today, in its visual, intuitive aspects. With the aid of visual imagination we can illuminate the manifold facts and problems of geometry, and beyond this, it is possible in many cases to depict the geometric outline of methods of investigation and proof (Hilbert & Cohn-Vossen (1952), iii).

The diagrams that Hilbert and Cohn-Vossen present in this book are meant to familiarize the reader with not only the facts and problems of geometry, but also to familiarize them with the "methods of investigation". As a result, these diagrams are playing a crucial role in explaining aspects of geometry to the reader. The fact that an entire book, co-written by one of the most prominent and successful mathematicians of the 20th century, is dedicated to presenting and discussing geometric diagrams is telling of the significance of diagrams in mathematical practice. Moreover, the comments made by Hilbert in the preface of this book indicate that what diagrams are doing is familiarizing us with the subject matter of geometry in a way that answers the "what's going on here" question and improves our geometric understanding.

The goal of this discussion was to highlight the breadth and significance of mathematical explanation. We see that mathematical explanation is not limited to the case of proof, but instead that things like definitions and diagrams can provide explanations and consequently contribute to our mathematical understanding.

4. Conclusion

This paper aimed to identify some of the advantages of accepting an account of mathematical explanation that is closely related to understanding. I began by proposing that we think of mathematical explanation in terms of answers to the question "what's going on here?" as this question is more likely to produce understanding than traditional why-questions are. This shift allows us to recognize the multi-facted nature of mathematical practice so that we can more accurately consider how explanation arises within mathematics. We see that the picture of mathematical explanation simply as explanatory proof oversimplifies the practice of mathematics and that instead explanations can come in forms of definitions and diagrams. Thus, the view that explanation is meant to generate understanding provides new insight into what might count as an explanation.

Notes

[1]Wilkenfeld's account of functional explanation can be found in Wilkenfeld (2014).

[2]See Steiner (1978), Kitcher (1981), Kitcher (1989), and Lange (2017) for the details of these accounts.

[3]An interested reader to the following articles that comment on these accounts: D'Alessandro (2019), Hafner & Mancosu (2005), Hamami & Morris (2020), Inglis & Mejía-Ramos (2019).

[4]It's worth noting that some authors who have written about explanatory proofs have acknowledge that explanatory proofs is not all there is to mathematical explanation. For instance, Steiner (1978), Lange (2017), and Lange (2018) acknowledge that there is more than just explanatory proof.

[5]Skow (2015) has argued that explanations (i.e., answers to why-questions) do not generally produce understanding. I do not mean to endorse this more general claim, but am instead restricting my claim to the domain of mathematics. As a result, I am suggesting that there may be something about mathematical explanation that makes it significantly different from explanation in other domains.

[6]See Crowe (1988) and Kline (1972) for discussions of the introduction of imaginary numbers and the initial skeptical response from the mathematical community.

[7]See Baumberger (2014) for a clear discussion of some of the different kinds of understanding.

[8]For a related discussion, see de Regt (2017) (ch 3) which argues that a pluralist view of scientific understanding is needed.

[9]Note that if a function is differentiable then it is also continuous, and so the belief that if a function is continuous then it is differentiable amounts to a conflation of the two notions.

[10]See Manheim (1964), chapter 4, for a detailed discussion of the discovery of these functions and the functions themselves. There is also a detailed discussion of the work being done in the 19th century to improve our understanding of functions and their properties in Kline (1972), chapter 40.

[11]These are the clearest ways that a function can be continuous but non-differentiable at a point. But there is another way that a function can be continuous and non-differentiable at a point. Specifically, this will happen when a function has a vertical tangent line at a point.

[12]See Giaquinto (2011) for a discussion of the visualizability of nowhere differentiable continuous functions and their failure to be "pencil continuous".

[13]Figure 5 was created by Sandy Ganzell and is used here with his permission.

[14]It is worth noting that the diagram in Figure 6 corresponds to the virtual trefoil knot. But this is not the standard representation of the virtual trefoil knot, though it is equivalent and so can be manipulated into the standard presentation by means of a series of virtual and classical Reidemeister moves.

[15]An example of such an invariant is the forbidden number, see Crans, Ganzell, & Mellor (2015).

[16]Some other authors have also considered mathematical explanations that appear in forms other than proof. See D'Alessandro (2020), Lange (2018), Lehet (2021).

[17]This account of explanatory definitions was first introduced in Lehet (2021).

[18]This is a fact that we used earlier when proving that for prime natural number, p, $\sqrt{p}$ is irrational.

[19]For the sake of simplicity, I am limiting the discussion to circles with centers at the origin.

[20]To clarify, these definitions are logically equivalent when the first definition is restricted to taking the center of the circle to be the origin or when the second definition is stated in full generality.

[21]Or at least, we come across approximations of circles that are indistinguishable from the real thing to the human eye.

[22]Again, for the sake of simplicity, I am assuming that the center of the ellipse is the origin.

[23]See Brown (2008), De Toffoli (2017), De Toffoli & Giardino (2014), De Toffoli & Giardino (2016), Mancosu (2008) for discussions of the epistemic significance of diagrams. Carter (2019) also provides an overview of what philosophical work has been done regarding the use of diagrams in mathematics.

[24]It is worth noting that this diagram is a variation of the one given in Steiner (1978), which uses two isosceles triangles to form a square rather than a rectangle. A square can be formed when we let the hypotenuses of the two triangles overlap.

[25]Here I have only mentioned two examples of texts that have focused on a diagrammatic presentation of material. As Mancosu (2008) points out, the late 20th century produced many other examples of texts that take a visual approach to mathematics.

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